Artificial Intelligence 1: Constraint Satisfaction problems

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Outline

- CSP?
- Backtracking for CSP
- Local search for CSPs
- Problem structure and decomposition

Constraint satisfaction problems

What is a CSP?
- Finite set of variables $V_1, V_2, \ldots, V_n$
- Finite set of constraints $C_1, C_2, \ldots, C_m$
- Nonemtpy domain of possible values for each variable $D_{V1}, D_{V2}, \ldots D_{Vn}$
- Each constraint C_i limits the values that variables can take, e.g., $V_1 \neq V_2$

A *state* is defined as an *assignment* of values to some or all variables.

Consistent assignment: assignment does not not violate the constraints.

Constraint satisfaction problems

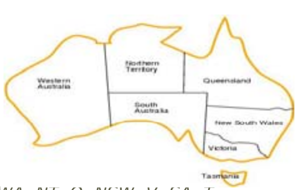
An assignment is *complete* when every value is mentioned.

A *solution* to a CSP is a complete assignment that satisfies all constraints.

Some CSPs require a solution that maximizes an *objective function*.

Applications: Scheduling the time of observations on the Hubble Space Telescope, Floor planning, Map coloring, Cryptography

CSP example: map coloring



Variables: *WA, NT, Q, NSW, V, SA, T*

Domains: D_i=*{red,green,blue}*

Constraints: adjacent regions must have different colors.
- E.g. $WA \neq NT$ (if the language allows this)
- E.g. $(WA,NT) \neq$ *{(red,green),(red,blue),(green,red),…}*

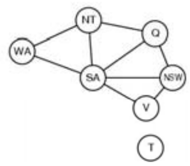
CSP example: map coloring

Solutions are assignments satisfying all constraints, e.g.

{WA=red,NT=green,Q=red,NSW=green,V=red,SA=blue,T=green}

Constraint graph

CSP benefits
- Standard representation pattern
- Generic goal and successor functions
- Generic heuristics (no domain specific expertise).



Constraint graph = nodes are variables, edges show constraints.
Graph can be used to simplify search.
e.g. Tasmania is an independent subproblem.

Varieties of CSPs

Discrete variables
- Finite domains; size $d \Rightarrow O(d^n)$ complete assignments.
  - E.g. Boolean CSPs, include. Boolean satisfiability (NP-complete).
- Infinite domains (integers, strings, etc.)
  - E.g. job scheduling, variables are start/end days for each job
  - Need a constraint language e.g $StartJob_1 + 5 \leq StartJob_3$.
  - Linear constraints solvable, nonlinear undecidable.

Continuous variables
- e.g. start/end times for Hubble Telescope observations.
- Linear constraints solvable in poly time by LP methods.

Varieties of constraints

Unary constraints involve a single variable.
- e.g. *SA ≠ green*

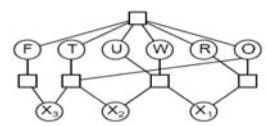
Binary constraints involve pairs of variables.
- e.g. *SA ≠ WA*

Higher-order constraints involve 3 or more variables.
- e.g. cryptharithmetic column constraints.

Preference (soft constraints) e.g. *red* is better than *green* often representable by a cost for each variable assignment → constrained optimization problems.

Example; cryptharithmetic

```
  T W O
+ T W O
F O U R
```



Variables: $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$
Domains: $\{0,1,2,3,4,5,6,7,8,9\}$
Constraints
alldiff(F,T,U,W,R,O)
$O+O=R+10\cdot X_1$, etc.

CSP as a standard search problem

A CSP can easily expressed as a standard search problem.

Incremental formulation
- *Initial State*: the empty assignment {}.
- *Successor function*: Assign value to unassigned variable provided that there is not conflict.
- *Goal test*: the current assignment is complete.
- *Path cost*: as constant cost for every step.

CSP as a standard search problem

This is the same for all CSP's !!!

Solution is found at depth n (if there are n variables).
- Hence depth first search can be used.

Path is irrelevant, so complete state representation can also be used.

Branching factor b at the top level is nd.

$b=(n-l)d$ at depth l, hence $n!d^n$ leaves (only d^n complete assignments).

Commutativity

CSPs are commutative.

– The order of any given set of actions has no effect on the outcome.

– Example: choose colors for Australian territories one at a time

  – [WA=red then NT=green] same as [NT=green then WA=red]

  – All CSP search algorithms consider a single variable assignment at a time ⇒ there are d^n leaves.

Backtracking search

Cfr. Depth-first search

Chooses values for one variable at a time and backtracks when a variable has no legal values left to assign.

Uninformed algorithm

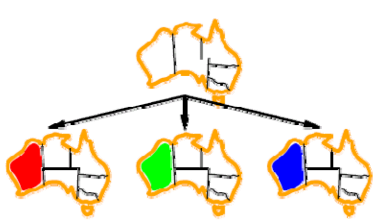
– No good general performance (see table p. 143)

Backtracking search

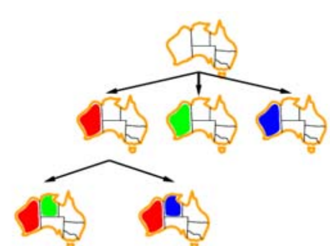
```
function BACKTRACKING-SEARCH(csp) return a solution or failure
    return RECURSIVE-BACKTRACKING({} , csp)

function RECURSIVE-BACKTRACKING(assignment, csp) return a solution or failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp],assignment,csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
          if value is consistent with assignment according to CONSTRAINTS[csp] then
                add {var=value} to assignment
                result ← RRECURSIVE-BACTRACKING(assignment, csp)
                if result ≠ failure then return result
                remove {var=value} from assignment
    return failure
```


Backtracking example

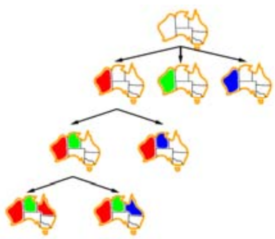
Backtracking example



Backtracking example



Backtracking example



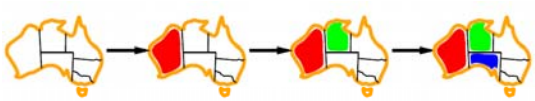
Improving backtracking efficiency

Previous improvements → introduce heuristics

General-purpose methods can give huge gains in speed:

- Which variable should be assigned next?
- In what order should its values be tried?
- Can we detect inevitable failure early?
- Can we take advantage of problem structure?

Minimum remaining values



var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[*csp*],*assignment*,*csp*)

A.k.a. most constrained variable heuristic

Rule: choose variable with the fewest legal moves

Which variable shall we try first?

Degree heuristic

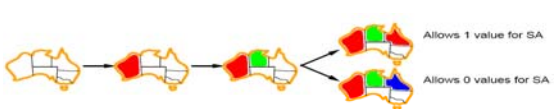
Use degree heuristic

Rule: select variable that is involved in the largest number of constraints on other unassigned variables.

Degree heuristic is very useful as a tie breaker.

In what order should its values be tried?

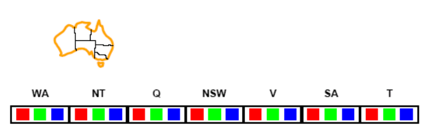
Least constraining value



Least constraining value heuristic

Rule: given a variable choose the least constraing value i.e. the one that leaves the maximum flexibility for subsequent variable assignments.

Forward checking



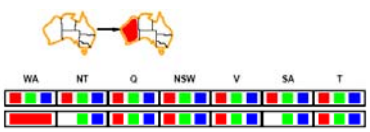
Can we detect inevitable failure early?

- *And avoid it later?*

Forward checking idea: keep track of remaining legal values for unassigned variables.

Terminate search when any variable has no legal values.

Forward checking

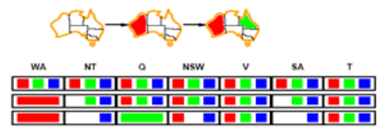


Assign *{WA=red}*

Effects on other variables connected by constraints with WA

- *NT can no longer be red*
- *SA can no longer be red*

Forward checking



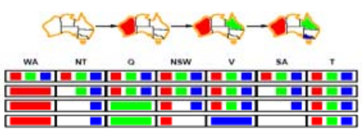
Assign *{Q=green}*

Effects on other variables connected by constraints with WA

- *NT can no longer be green*
- *NSW can no longer be green*
- *SA can no longer be green*

MRV heuristic will automatically select NT and SA next, why?

Forward checking



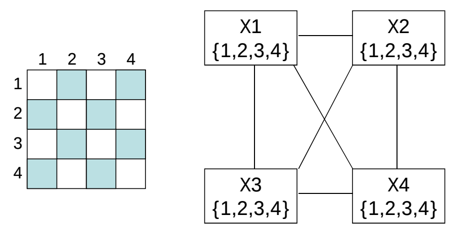
If *V* is assigned *blue*

Effects on other variables connected by constraints with WA

- *SA is empty*
- *NSW can no longer be blue*

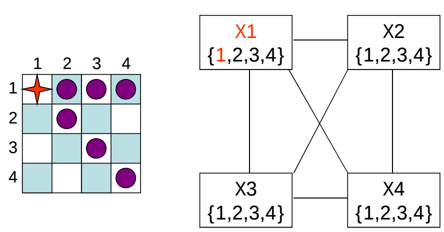
FC has detected that partial assignment is *inconsistent* with the constraints and backtracking can occur.

Example: 4-Queens Problem



[4-Queens slides copied from B.J. Dorr CMSC 421 course on AI]

Example: 4-Queens Problem



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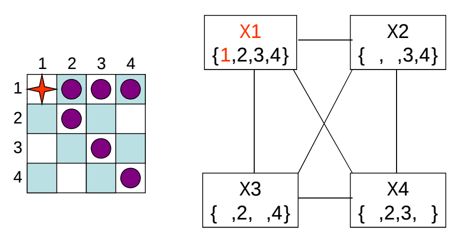
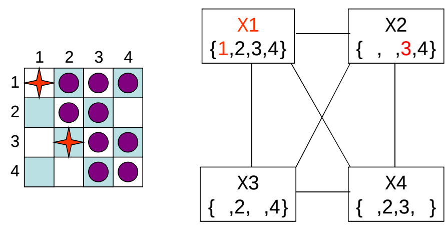
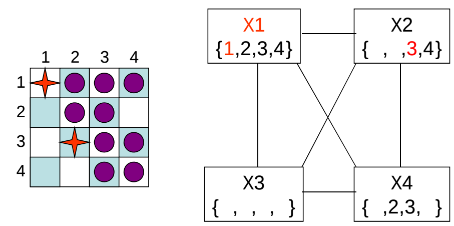


Example: 4-Queens Problem



Example: 4-Queens Problem



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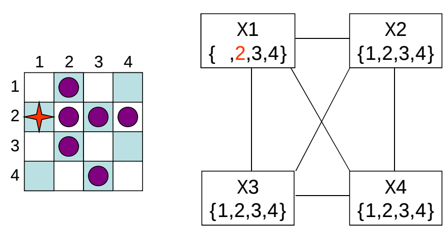
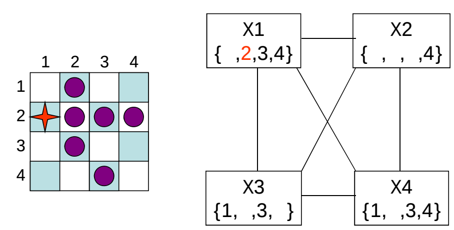


Example: 4-Queens Problem



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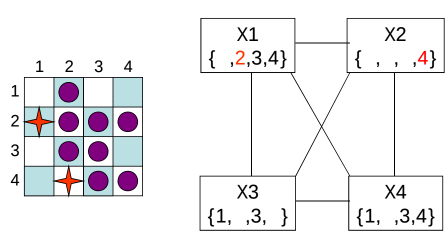

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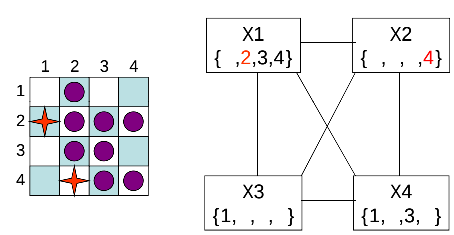

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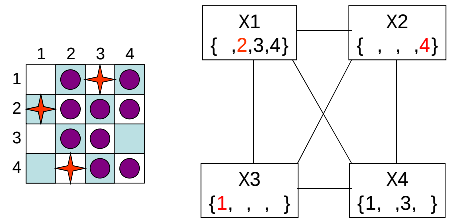

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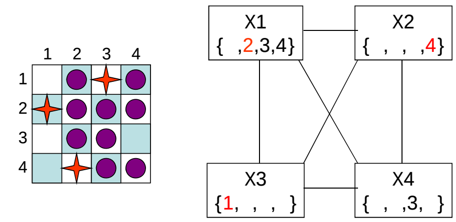
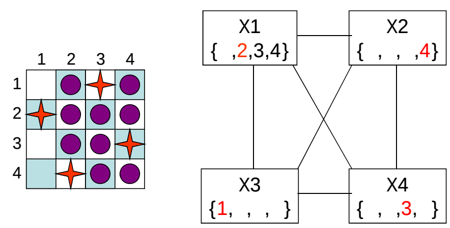
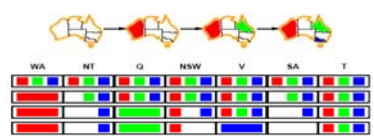


Example: 4-Queens Problem



Constraint propagation



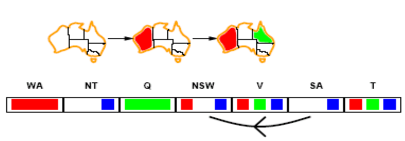
Solving CSPs with combination of heuristics plus forward checking is more efficient than either approach alone.

FC checking propagates information from assigned to unassigned variables but does not provide detection for all failures.

- NT and SA cannot be blue!

Constraint propagation repeatedly enforces constraints locally

Arc consistency



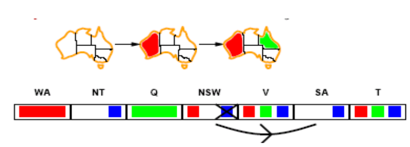
X → *Y* is consistent iff

  for *every* value *x* of *X* there is some allowed *y*

SA → *NSW* is consistent iff

  SA=blue and *NSW=red*

Arc consistency



X → *Y* is consistent iff

  for *every* value *x* of *X* there is some allowed *y*

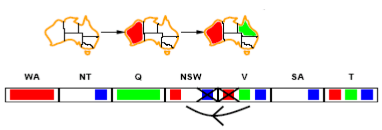
NSW → *SA* is consistent iff

  NSW=red and *SA=blue*

  NSW=blue and SA=???

Arc can be made consistent by removing *blue* from *NSW*

Arc consistency

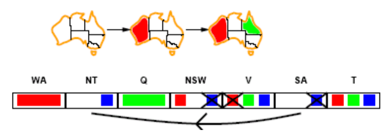


Arc can be made consistent by removing *blue* from *NSW*

RECHECK neighbours !!

– Remove red from *V*

Arc consistency



Arc can be made consistent by removing *blue* from *NSW*

RECHECK neighbours !!

– Remove red from *V*

Arc consistency detects failure earlier than FC

Can be run as a preprocessor or after each assignment.

– Repeated until no inconsistency remains

Arc consistency algorithm

function AC-3(*csp*) **return** the CSP, possibly with reduced domains
    inputs: *csp*, a binary csp with variables $\{X_1, X_2, \ldots, X_n\}$
    local variables: *queue,* a queue of arcs initially the arcs in *csp*

    while queue is not empty **do**
        $(X_i, X_j) \leftarrow$ REMOVE-FIRST(*queue*)
        if REMOVE-INCONSISTENT-VALUES(X_i, X_j) **then**
            for each X_k **in** NEIGHBORS[X_i] **do**
                add (X_i, X_j) to queue

function REMOVE-INCONSISTENT-VALUES(X_i, X_j) **return** *true* iff we remove a value
    removed $\leftarrow$ *false*
    for each *x* in DOMAIN[X_i] **do**
        if no value *y* in DOMAIN[X_j] allows (x,y) to satisfy the constraints between X_i and X_j
        then delete x from DOMAIN[X_i]; *removed* $\leftarrow$ *true*
    return *removed*

K-consistency

Arc consistency does not detect all inconsistencies:

– Partial assignment *{WA=red, NSW=red}* is inconsistent.

Stronger forms of propagation can be defined using the notion of k-consistency.

A CSP is k-consistent if for any set of k-1 variables and for any consistent assignment to those variables, a consistent value can always be assigned to any kth variable.

– E.g. 1-consistency or node-consistency
– E.g. 2-consistency or arc-consistency
– E.g. 3-consistency or path-consistency

K-consistency

A graph is strongly k-consistent if

– It is k-consistent and
– Is also (k-1) consistent, (k-2) consistent, … all the way down to 1-consistent.

This is ideal since a solution can be found in time $O(nd)$ instead of $O(n^2d^3)$

YET *no free lunch*: any algorithm for establishing n-consistency must take time exponential in n, in the worst case.

Further improvements

Checking special constraints

– Checking Alldif(…) constraint
    – *E.g. {WA=red, NSW=red}*
– Checking Atmost(…) constraint
    – *Bounds propagation for larger value domains*

Intelligent backtracking

– Standard form is chronological backtracking i.e. try different value for preceding variable.
– More intelligent, backtrack to conflict set.
    – Set of variables that caused the failure or set of previously assigned variables that are connected to X by constraints.
    – Backjumping moves back to most recent element of the conflict set.
    – Forward checking can be used to determine conflict set.

Local search for CSP

Use complete-state representation

For CSPs
- allow states with unsatisfied constraints
- operators **reassign** variable values

Variable selection: randomly select any conflicted variable

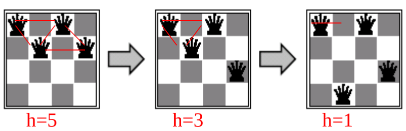
Value selection: *min-conflicts heuristic*
- Select new value that results in a minimum number of conflicts with the other variables

Local search for CSP

function MIN-CONFLICTS(*csp, max_steps*) **return** solution or failure
inputs: *csp*, a constraint satisfaction problem
max_steps, the number of steps allowed before giving up

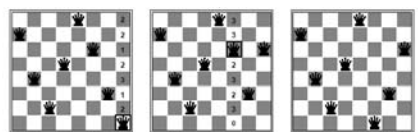
current ← an initial complete assignment for *csp*
for *i* = 1 to *max_steps* **do**
if *current* is a solution for *csp* then return *current*
var ← a randomly chosen, conflicted variable from VARIABLES[*csp*]
value ← the value *v* for *var* that minimize CONFLICTS(*var,v,current,csp*)
set *var* = *value* in *current*
return *failure*

Min-conflicts example 1



Use of min-conflicts heuristic in hill-climbing.

Min-conflicts example 2



A two-step solution for an 8-queens problem using min-conflicts heuristic.

At each stage a queen is chosen for reassignment in its column.

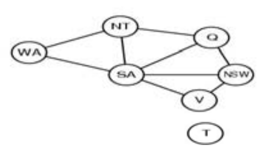
The algorithm moves the queen to the min-conflict square breaking ties randomly.

Advantages of local search

The runtime of min-conflicts is roughly independent of problem size.
- Solving the millions-queen problem in roughly 50 steps.

Local search can be used in an online setting.
- Backtrack search requires more time

Problem structure

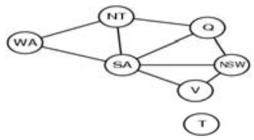


How can the problem structure help to find a solution quickly?

Subproblem identification is important:
- Coloring Tasmania and mainland are independent subproblems
- Identifiable as connected components of constrained graph.

Improves performance

Problem structure



Suppose each problem has *c* variables out of a total of *n*.

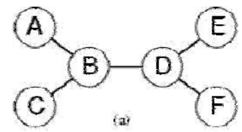
Worst case solution cost is $O(n/c\ d^c)$, i.e. linear in *n*

- Instead of $O(d^n)$, exponential in *n*

E.g. $n= 80,\ c= 20,\ d=2$

- 2^{80} = 4 billion years at 1 million nodes/sec.
- 4 * 2^{20}= .4 second at 1 million nodes/sec

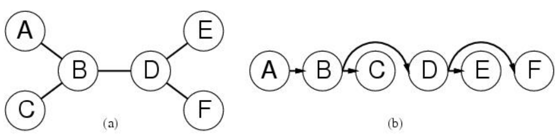
Tree-structured CSPs



Theorem: if the constraint graph has no loops then CSP can be solved in $O(nd^2)$ time

Compare difference with general CSP, where worst case is $O(d^n)$

Tree-structured CSPs

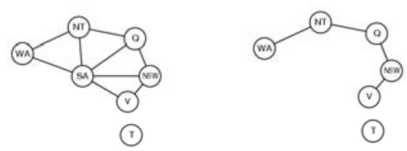


In most cases subproblems of a CSP are connected as a tree

Any tree-structured CSP can be solved in time linear in the number of variables.

- Choose a variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering. (label var from X_1 to X_n)
- For *j* from *n* down to 2, apply REMOVE-INCONSISTENT-VALUES(Parent(X_j),X_j)
- For *j* from 1 to *n* assign X_j consistently with Parent(X_j)

Nearly tree-structured CSPs

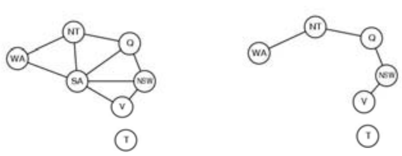


Can more general constraint graphs be reduced to trees?

Two approaches:

- Remove certain nodes
- Collapse certain nodes

Nearly tree-structured CSPs

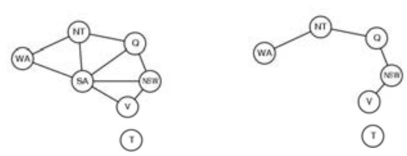


Idea: assign values to some variables so that the remaining variables form a tree.

Assume that we assign *{SA=x}* ← *cycle cutset*

- And remove any values from the other variables that are inconsistent.
- The selected value for SA could be the wrong one so we have to try all of them

Nearly tree-structured CSPs



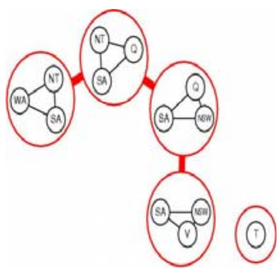
This approach is worthwhile if cycle cutset is small.

Finding the smallest cycle cutset is NP-hard

- Approximation algorithms exist

This approach is called *cutset conditioning*.

Nearly tree-structured CSPs



- Tree decomposition of the constraint graph in a set of connected subproblems.
- Each subproblem is solved independently
- Resulting solutions are combined.
- Necessary requirements:
  - Every variable appears in ar least one of the subproblems.
  - If two variables are connected in the original problem, they must appear together in at least one subproblem.
  - If a variable appears in two subproblems, it must appear in each node on the path.

Summary

- CSPs are a special kind of problem: states defined by values of a fixed set of variables, goal test defined by constraints on variable values
- Backtracking=depth-first search with one variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that lead to failure.
- Constraint propagation does additional work to constrain values and detect inconsistencies.
- The CSP representation allows analysis of problem structure.
- Tree structured CSPs can be solved in linear time.
- Iterative min-conflicts is usually effective in practice.

Taak 2: CSP and sudoku

http://arti.vub.ac.be/cursus/2005-2006/AI1/taak2/sudoku.htm

QuickTime™ and a
TIFF (LZW) decompressor
are needed to see this picture.

- Variabelen?
  - 81 vakjes
- Domeinen?
  - Tussen 1-9
- Constraints?
  - Tussen elke twee variabelen in dezelfde rij, kolom en 3x3 group.

Taak 2: CSP and sudoku

QuickTime™ and a
TIFF (LZW) decompressor
are needed to see this picture.

- DEEL 1
  - CSP oplossen met forward checking en MGU heuristiek.
- DEEL 2
  - Local search op basis van Evolutionary transition algorithm (ETA).
- Schrijf verslag.